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Application of a finite element method for variational inequalities

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Abstract

In this paper we explore the application of a finite element method (FEM) to the inequality and Laplacian constrained variational optimization problems. First, we illustrate the connection between the optimization problem and elliptic variational inequalities; secondly, we prove the existence of the solution via the augmented Lagrangian multipliers method. A triangular type FEM is employed in the numerical calculations. Computational results indicate that the present finite element method is a highly efficient technique in these sorts of variational problems involving inequalities.

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1 Introduction

Nonlinear variational inequalities (VI) and optimal control problems are a significantly important class of problems having a broad range of applications in mathematics, engineering, mechanics and physics. Elasticity, computational fluid mechanics, computational mechanics are some of the particular applications of these types of problems.

In the present day, nonlinear variational problems or partial differential equation based optimal control problems are not only important but also very challenging problems which usually involve high storage requirements, hard optimization and solution techniques. Therefore, finding reliable and efficient computational and numerical techniques along with fast implementation methods for the solution of these types of problems is quite useful and one of active research areas in the subject. In this paper we explore the application of a finite element method (FEM) [1] to the inequality and Laplacian constrained variational optimization problems [2].

Structure of this paper can be summarized as follows. Section 2 overviews some related methods briefly. In Section 3 we express the model problem as an inequality and Laplacian constrained variational optimization problem. In the same section, we present a theorem which shows the connection between the optimization problem and the second kind variational inequality problems. We discretize the VI problems using a finite element method. We complete the paper with a conclusion section where we discuss the method presented in this paper and point some possible future extensions.

2 Related studies

In this section our goal is to present some related work where some type of FEM is employed as a numerical solution technique. Although there are a vast amount of related studies, we will try to mention only a few of those which have some particular and significant application areas and consider the ones which set up the connection between variational inequalities and optimal control problems besides employing FEM in numerical calculations. Before we start mentioning some of those related studies, we first illustrate an application of the present FEM to a specific variational inequality problem. Let $V = H_0^1(\Omega)$, and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega,$$

where

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2},$$

and $L(v) = \langle f, v\rangle$ for $f \in V^* = H^{-1}(\Omega)$ and $v \in V$. Let $\Psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ and $\Psi_{|\Gamma} \leq 0$. Define $K = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. on } \Omega\}$. Then the variational inequality problem to which to present FEM is applied is defined as finding u such that

$$a(u, v-u) \geq L(v-u) \quad \text{for all } v \in K, u \in K.$$

Detailed information about this variational inequality problem and the application of the present FEM to this problem might be obtained from [3].

In [4] Barret *et al.* study the existence of a solution to an open problem about a quasi-variational inequality problem arising in a model for sand surface evolution. The authors first introduce a regularized mixed formulation involving both the primal and dual variables, and secondly study two methods for numerical solution of the problem. Detailed information about this study might be obtained in [4].

In [5] Burke *et al.* pose the Francfort-Marigo model of brittle fracture in terms of the minimization of a quite irregular energy functional. Employing a generalized Ambrosio-Tortorelli functional, the authors obtain a bound-constrained minimization problem, which can be expressed in terms of a variational inequality. By proposing a new FEM, the authors compute the local minimizer of the generalized functional.

In [6] Elliott *et al.* introduce a finite element approximation of a variational formulation of Bean's model for the physical configuration of an infinitely long cylindrical superconductor subject to a transverse magnetic field. The authors prove an error between the exact solution and the approximate solution for the current density and the magnetic field in appropriate norms. In the paper some numerical simulations for a variety of applied magnetic fields are also presented.

In [7] Bank *et al.* consider the application of primal-dual interior methods to the optimization of systems arising in the finite-element discretization of a class of elliptic variational inequalities. These problems lead to very large optimization problems with upper and lower bound constraints. When interior methods are applied to the discretized problem, the resulting linear systems have the same zero/nonzero structure as the finite-element equations solved for the unconstrained case.

As we have pointed above, although it is highly possible to extend this list of related studies, we refer the interested reader to the literature for further related work within the present paper.

3 Main results

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We consider a model problem defined as follows.

For given $f \in L^2(\Omega)$, find a scalar function ϕ on Ω being a minimizer of the cost functional

$$J(u) := \frac{1}{2}\int_\Omega \Delta u \Delta u \, d\Omega + \int_\Omega |\nabla u| \, d\Omega - \int_\Omega fu \, d\Omega \tag{1}$$

over the convex set K defined by

$$K = \left\{ u \in H_0^1(\Omega) : |\nabla u| \leq 1 \text{ almost everywhere on } \Omega \right\},$$

where the Sobolev space $H_0^1(\Omega)$ is defined as follows:

$$H^1(\Omega) = \left\{ \phi \in L^2(\Omega), \frac{\partial \phi}{\partial x_i} \in L^2(\Omega), \forall i = 1, \ldots, N \right\},$$
$$H_0^1(\Omega) = \left\{ \phi \in H^1(\Omega), \phi = 0 \text{ on } \partial\Omega \right\}.$$

Variational inequalities might be separated into two main groups as elliptic and parabolic variational inequalities. Glowinski studies these sorts of inequalities in [3] in detail. He considers the elliptic variational inequalities (EVI) as the first kind and second kind EVI and defines those in a functional context as follows.

Let V denote a real Hilbert space with the inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. V^* is a dual space of V, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear, continuous and V-elliptic form on $V \times V$, $L : V \to \mathbb{R}$ continuous linear functional, K is a closed convex nonempty subset of V, $j(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ is a convex lower semi-continuous functional. The first and second kind EVI are typically defined in the following way.

The first kind EVI: find $\phi \in V$ such that ϕ is a solution of the problem

$$a(\phi, v - \phi) \geq L(v - \phi), \quad \text{every } v \in K, \phi \in K.$$

The second kind EVI: find $\phi \in V$ such that ϕ is a solution of the problem

$$a(\phi, v - \phi) + j(v) - j(\phi) \geq L(v - \phi), \quad \text{every } v \in V, \phi \in V.$$

The next lemma sets up the connection between optimization and VI problems.

Lemma 1 [3] *Let $b : V \times V \to \mathbb{R}$ be a symmetric continuous bilinear V-elliptic form. Let $L \in V^*$ and $j : V \to \mathbb{R} \cup \{\infty\}$ be a convex lower semi-continuous proper functional. Let*

$$J(v) = \frac{1}{2}b(v, v) + j(v) - L(v).$$

Then the minimization problem, find ϕ such that

$$J(\phi) \leq J(v), \quad \text{every } v \in V, \phi \in V,$$

has a unique solution which is characterized by

$$b(\phi, v - \phi) + j(v) - j(\phi) \geq L(v - \phi), \quad \text{every } v \in V, \phi \in V.$$

Proof The proof can be seen on p.7 of [3]. □

It is clear that we can define the following in terms of the notations of Lemma 1:

$$b(u,u) := (\Delta u, \Delta u) = \int_\Omega \Delta u \Delta u \, d\Omega,$$
$$j(v) := \int_\Omega |\nabla v| \, d\Omega,$$
$$L(v) := \int_\Omega fv \, d\Omega.$$

We will approximate (1) with a finite element method introduced in [3]. Assume that Ω is a polygonal domain of $\mathbb{R}^2$. Consider a triangulation $\mathfrak{F}_h$ of Ω in the following sense: $\mathfrak{F}_h$ is a finite set of triangles T such that

$$T \subset \overline{\Omega} \quad \forall T \in \mathfrak{F}_h, \bigcup_{T \in \mathfrak{F}_h} T = \overline{\Omega},$$
$$T_1^\circ \cap T_2^\circ = \emptyset \quad \forall T_1, T_2 \in \mathfrak{F}_h \text{ and } T_1 \neq T_2.$$

Here T_i° denotes the inner part of the corresponding triangle. Furthermore, for all $T_1, T_2 \in \mathfrak{F}_h$ and $T_1 \neq T_2$, exactly one of the following conditions must hold:

(i) $T_1 \cap T_2 = \emptyset$,
(ii) T_1 and T_2 have only one common vertex,
(iii) T_1 and T_2 have only a whole common edge.

h is the length of the largest edge of the triangles in the triangulation. Define P_k as a space of polynomials in x_1 and x_2 of degree less than or equal to k, and

$$\sum_h := \{P \in \overline{\Omega}, P \text{ is a vertex of } T \in \mathfrak{F}_h\}.$$

The space $V = H_0^1(\Omega)$ is approximated by the family of subspaces $(V_h^k)_h$ with $k = 1$ or $k = 2$, where

$$V_h^k := \{v_h \in C^0(\overline{\Omega}), v_h|_{\partial\Omega} = 0 \text{ and } v_h|_T \in P_k, \forall T \in \mathfrak{F}_h\}, \quad k = 1, 2.$$

It is obvious that the V_h^k are finite dimensional. Then the space K is approximated by

$$K_h^k = \left\{ v_h \in V_h^k, v_h(P) \geq \psi(P), \forall P \in \sum_h^k \right\}, \quad k = 1, 2.$$

Notice that K_h^k for $k = 1, 2$ are closed convex nonempty subsets of V_h^k.

With these settings, the solution $\phi \in K$ is approximated by

$$b(\phi_h, u - \phi_h) + j(u) - j(\phi_h) \geq L(u - \phi_h), \quad \text{every } u \in K_h, \phi \in K_h, \tag{2}$$

or equivalently,

$$\begin{aligned} &\int_\Omega \bigl(\Delta\phi_h \Delta(u - \phi_h)\bigr)\, d\Omega + \int_\Omega |\nabla u|\, d\Omega - \int_\Omega |\nabla\phi_h|\, d\Omega \\ &\quad \geq \int_\Omega f(u - \phi_h)\, d\Omega, \quad \text{every } u \in K_h, \phi \in K_h. \end{aligned}$$

Using the augmented Lagrangian multipliers method, we find a discrete solution of (1) as follows. First, let us introduce the Lagrange functional

$$\widehat{\mathcal{L}}(u, \mu) = \frac{1}{2} b(u, u) + \int_\Omega |\nabla u|\, d\Omega - \int_\Omega fu\, d\Omega + \int_\Omega \mu\bigl(|\nabla u| - 1\bigr)\, d\Omega.$$

For $r \geq 0$, an augmented Lagrangian $\mathcal{L}_r$ is defined by

$$\mathcal{L}_r(u, \mu) = \widehat{\mathcal{L}}(u, \mu) + \frac{r}{2} \int_\Omega |\nabla u - 1|^2\, d\Omega. \tag{3}$$

Augmented Lagrangian multipliers methods for VI problems have been introduced by Glowinski and Marrocco (see [8]). Theorem 2.1 on p.168 in [3] guarantees the existence of a solution of this optimization problem.

Let us do notice that the first component ϕ_h of (2) is then the solution of the original problem (1). Using the techniques of the variational calculus, we can write that

$$\bigl((1 + \mu_h)\Delta\phi_h, \Delta(v - \phi_h)\bigr) + j(v) - j(\phi_h) - L(u - \phi_h) = \bigl(\mu_h, \Delta(v - \phi_h)\bigr) \quad \forall\phi \in V_h,$$

or equivalently,

$$\begin{aligned} &\int_\Omega \bigl((1 + \mu_h)\Delta\phi_h \Delta(v - \phi_h)\bigr)\, d\Omega + \int_\Omega |\nabla u|\, d\Omega - \int_\Omega |\nabla\phi_h|\, d\Omega - \int_\Omega f(u - \phi_h)\, d\Omega \\ &\quad = \int_\Omega \mu_h \Delta(v - \phi_h)\, d\Omega. \end{aligned}$$

We can describe the solution algorithm as follows:

(i) Choose an initial iterate μ_h and $\lambda \geq 0$;

(ii) Solve the linear problem
$((1 + \mu_h)\Delta\phi_h, \Delta(v - \phi_h)) + j(v) - j(\phi_h) - L(u - \phi_h) = (\mu_h, \Delta(v - \phi_h)), \forall\phi \in V_h$;

(iii) Update $\mu_h^{\alpha+1} = \max\{0, \mu_h + \alpha(\nabla u_h^\alpha - 1)\}$ on each cell;

(iv) Set $\alpha = \alpha + 1$ and go back to (ii).

4 Conclusion

In this note we have applied a finite element method to the inequality and Laplacian constrained variational optimization problems. Firstly, we illustrate the connection between the optimization problem and elliptic variational inequalities; and secondly, we solve the problem using the augmented Lagrangian multipliers method. In a future work, we are planning to apply the present method to some specific application areas including modeling with optimal control problems.

Competing interests
The author declares that they have no competing interests.

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